MATH2050C Selected Solution to Assignment 7

Section 3.7

(11). Yes, $\sum a_n^2$ is convergent when $\sum a_n$ is convergent where $a_n \geq 0$. For, when the latter series is convergent, it implies in particular that $\{a_n\}$ is bounded. We can find some M such that $0 \leq a_n < M$. For $\varepsilon > 0$, there exists some n_0 such that $\sum_{k=m+1}^{n} a_n < \varepsilon/M$ for all $n, m \geq n_0$. But then

$$\sum_{k=m+1}^{n} a_k^2 \leq M \sum_{k=m+1}^{n} a_k < M\frac{\varepsilon}{M} = \varepsilon \ ,$$

so $\sum a_n^2$ is convergent by Cauchy Convergence Criterion.

(12). No. It suffices to consider $\sum 1/n^2$.

(15). Use induction to show

$$\frac{1}{2}(a(1) + 2a(2) + \cdots + 2^n a(2^n)) \leq s(2^n) \leq (a(1) + 2a(2) + \cdots + 2^{n-1}a(2^{n-1})) + a(2^n) \ ,$$

where $a_n > 0$ is decreasing. We work out the right inequality and leave the left one to you. When $n = 1$, the right inequality becomes

$$a(1) + a(2) \leq a(1) + a(2),$$

which is trivial. Assume it is true for n and we establish it for $n + 1$. Indeed, by induction hypothesis and the fact that $\{a_n\}$ is decreasing,

$$\begin{aligned}
s(2^{n+1}) &= a(1) + a(2) + \cdots + a(2^n) + a(2^n + 1) + \cdots + a(2^{n+1}) \\
&= s(2^n) + a(2^n + 1) + \cdots + a(2^{n+1}) \\
&\leq \big(a(1) + \cdots + 2^{n-1}a(2^{n-1}) + a(2^n)\big) + a(2^n + 1) + \cdots + a(2^{n+1}) \\
&= a(1) + \cdots + 2^{n-1}a(2^{n-1}) + \big(a(2^n) + a(2^n + 1) + \cdots + a(2^{n+1} - 1)\big) + a(2^{n+1}) \\
&\leq a(1) + \cdots + 2^{n-1}a(2^{n-1}) + 2^n a(2^n) + a(2^{n+1}) \ ,
\end{aligned}$$

done.

(16). We look at $\sum_{n=1}^{\infty} 2^n a(2^n) = \sum_{n=1}^{\infty} 2^n/2^{np} = \sum_{n=1}^{\infty} 2^{(1-p)n}$, which is convergent if and only if $p > 1$. We conclude that the p-series is convergent if and only if $p > 1$.

Supplementary Exercises

1. Show that for any sequence, there associates an infinite series whose convergence/divergence is the same as that for the sequence.

   Solution. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Define an infinite series by $\sum_{n=1}^{\infty} x_n$ where $x_1 = a_1, x_2 = -a_1 + a_2, x_3 = -a_2 + a_3, \cdots, x_n = -a_{n-1} + a_n$. Then the n-th partial sum of this series is equal to a_n, so the convergence/divergence of this series is the same as that of $\{a_n\}$.

2. An infinite series $\sum_n a_n$ is called **absolutely convergent** if $\sum_n |a_n|$ is convergent. Show that an absolutely convergent infinite series is convergent but the convergence is not always true.

Solution. By Cauchy Convergence Criterion, when $\sum |a_n|$ is convergent, for each $\varepsilon > 0$, there is some n_0 such that

$$\sum_{k=m+1}^{n} |a_k| < \varepsilon, \quad \forall n, m \geq n_0.$$

But then by the triangle inequality it implies

$$\left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| < \varepsilon, \quad \forall n, m \geq n_0 \ ,$$

in other words, the sequence of partial sums for $\sum a_n$ forms a Cauchy sequence and hence is convergent.

The series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ is convergent but $\sum_{n=1}^{\infty} 1/n$ is divergent.